

Stationary behaviour of observables after a quantum quench in the spin-1/2 Heisenberg XXZ chain

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We consider a quantum quench in the spin-1/2 Heisenberg XXZ chain. At late times after the quench it is believed that the expectation values of local operators approach time-independent values, that are described by a generalized Gibbs ensemble. Employing a quantum transfer matrix approach we show how to determine short-range correlation functions in such generalized Gibbs ensembles for a class of initial states.

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I. INTRODUCTION

Nonequilibrium dynamics in closed quantum systems, and in particular quantum quenches, have attracted much experimental^{1–6} and theoretical^{7–41} attention in recent years. There is a growing consensus that integrable models exhibit important differences in behaviour as compared to non-integrable ones⁴². In particular, by now there is ample evidence that the stationary state after a quantum quench in an integrable theory is described by a generalized Gibbs ensemble (GGE)⁸ with density matrix

$$\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} \exp \left(- \sum_{l=1} \lambda_l H^{(l)} \right). \quad (1.1)$$

Here $H^{(1)}$ is the Hamiltonian and $H^{(l)}$ are local²¹ integrals of motion

$$[H^{(m)}, H^{(n)}] = 0. \quad (1.2)$$

The Lagrange multipliers λ_l are fixed by the requirement that the expectation values of the integrals of motion are time-independent

$$\lim_{L \rightarrow \infty} \frac{\langle \Psi_0 | H^{(l)} | \Psi_0 \rangle}{L} = \lim_{L \rightarrow \infty} \frac{\text{Tr} [\rho_{\text{GGE}} H^{(l)}]}{L}. \quad (1.3)$$

Here L is the size of the system under consideration. In practice it is often useful to work with a truncated GGE³⁹, where only the y “most local” conservation laws are retained

$$\rho_{\text{tGGE}}^{(y)} = \frac{1}{Z_{\text{tGGE}}^{(y)}} \exp \left(- \sum_{l=1}^y \lambda_l^{(y)} H^{(l)} \right). \quad (1.4)$$

Here the $\lambda_l^{(y)}$ are fixed by

$$\lim_{L \rightarrow \infty} \frac{\langle \Psi_0 | H^{(l)} | \Psi_0 \rangle}{L} = \lim_{L \rightarrow \infty} \frac{\text{Tr} [\rho_{\text{tGGE}}^{(y)} H^{(l)}]}{L}, \quad l = 1, \dots, y. \quad (1.5)$$

The full GGE is then recovered in the limit $y \rightarrow \infty$, after the thermodynamic limit has been taken first. Assuming that a given integrable system indeed approaches a stationary state late after a quantum quench, which is described by a generalized Gibbs ensemble, important questions are how to construct the GGE in practice, and how to then determine expectation values of local operators. It is these questions we aim to address for the particular case of the spin-1/2 Heisenberg XXZ chain. A priori there are four steps:

1. Determine the local conservation laws.
2. Calculate their expectation values in the initial state after the quench.
3. Construct the GGE density matrix in such a way that equations (1.3) are fulfilled.
4. Determine the expectation values of local operators in this ensemble.

In the following we address these in turn. As local conservation laws we consider the *minimal set* obtained from the logarithmic derivative of the transfer matrix at the “shift-point”⁴³. It has been recently found that the XXZ Hamiltonian in general has local conservation laws that are not obtained in this way⁴⁴. In principle they could be accommodated in our construction as well. However, in order to keep things simple, we restrict our analysis to the antiferromagnetically ordered regime of the Heisenberg chain, where, as far as we know, the minimal set of local conservation laws is complete. With regards to step 2, we focus on a class of simple quenches, for which the initial states are unentangled. We show how to treat these cases analytically. Our method generalizes to weakly entangled initial states of matrix product form, but the analysis becomes much more complicated. The GGE density matrix is constructed by the quantum transfer matrix method⁴⁵. The most difficult issue here is what values the Lagrange multipliers λ_j take. We argue that it is possible to completely specify the quantum transfer matrix, without having to explicitly calculate the λ_j . Finally, GGE expectation values of local operators can be calculated by borrowing the results of the Wuppertal group for finite temperature correlators^{46,47}.

II. LOCAL INTEGRALS OF MOTION

We consider the XXZ Hamiltonian

$$H^{(1)} = \frac{1}{4} \sum_{\ell=1}^L \sigma_{\ell}^x \sigma_{\ell+1}^x + \sigma_{\ell}^y \sigma_{\ell+1}^y + \Delta (\sigma_{\ell}^z \sigma_{\ell+1}^z - 1), \quad (2.1)$$

where L is even, σ_j^{α} are Pauli matrices ($\sigma_{L+1}^{\alpha} \equiv \sigma_1^{\alpha}$) and we parametrize the anisotropy as

$$\Delta = \cos \gamma. \quad (2.2)$$

It is well known that (2.1) is solvable by the algebraic Bethe Ansatz method⁴³. Local conservation laws $H^{(k)}$ can then be obtained from the logarithmic derivative of the transfer matrix $\tau(\lambda)$

$$H^{(k)} = i \left(\frac{\sin \gamma}{\gamma} \frac{\partial}{\partial \lambda} \right)^k \log \tau(i + \lambda) \Big|_{\lambda=0}. \quad (2.3)$$

By definition the conservation laws commute with one another

$$[H^{(k)}, H^{(n)}] = 0. \quad (2.4)$$

The transfer matrix is constructed by Algebraic Bethe Ansatz and takes the form

$$\begin{aligned} \tau(i + \lambda) &= \text{Tr} [L_L(\lambda) L_{L-1}(\lambda) \dots L_1(\lambda)], \\ L_j(\lambda) &= \frac{1 + \tau^z \sigma_j^z}{2} + \frac{\sinh(\frac{\gamma\lambda}{2})}{\sinh(i\gamma + \frac{\gamma\lambda}{2})} \frac{1 - \tau^z \sigma_j^z}{2} + \frac{\sinh(i\gamma)}{\sinh(i\gamma + \frac{\gamma\lambda}{2})} (\tau^+ \sigma_j^- + \tau^- \sigma_j^+), \end{aligned} \quad (2.5)$$

where $\tau^{x,y,z}$ are Pauli matrices acting on the auxiliary space, and the trace is taken over the latter. In the following we denote indices in the auxiliary and quantum spaces by Roman (a, b) and Greek (α, β) letters respectively.

III. EXPECTATION VALUES OF LOCAL INTEGRALS OF MOTION IN THE INITIAL STATE

Given an initial state $|\Psi_0\rangle$, we aim to determine the expectation values

$$\langle \Psi_0 | H^{(k)} | \Psi_0 \rangle. \quad (3.1)$$

It is convenient to work with the generating function

$$\Omega_{\Psi_0}(\lambda) = \frac{1}{L} \langle \Psi_0 | \tau'(i + \lambda) \tau^{-1}(i + \lambda) | \Psi_0 \rangle = -i \sum_{k=1} \left(\frac{\gamma}{\sin \gamma} \right)^k \frac{\lambda^{k-1}}{(k-1)!} \frac{\langle \Psi_0 | H^{(k)} | \Psi_0 \rangle}{L}, \quad (3.2)$$

where the right hand side follows from (2.3). In order to evaluate $\Omega_{\Psi_0}(\lambda)$ we use that, when viewed as a power series in λ for large L , we have formally

$$\tau(i + \lambda) \sim \tau(i) \exp \left(-i \sum_{k=1} \left(\frac{\gamma}{\sin \gamma} \right)^k \frac{\lambda^k}{k!} H^{(k)} \right). \quad (3.3)$$

This suggests that for large L we have

$$\tau^{-1}(i + \lambda) = [\tau(i + \lambda)]^\dagger, \quad (3.4)$$

in the sense that the power-series expansions in λ coincide. These observations lead to the following (approximate) expression for the inverse

$$\tau^{-1}(i + \lambda) \sim \text{Tr}[M_L(\lambda)M_{L-1}(\lambda)\dots M_1(\lambda)], \quad (3.5)$$

$$M_j(\lambda) = \frac{1 + \tau^z \sigma_j^z}{2} + \frac{\sinh(\frac{\gamma^* \lambda}{2})}{\sinh(-i\gamma^* + \frac{\gamma^* \lambda}{2})} \frac{1 - \tau^z \sigma_j^z}{2} + \frac{\sinh(-i\gamma^*)}{\sinh(-i\gamma^* + \frac{\gamma^* \lambda}{2})} (\tau^+ \sigma_j^+ + \tau^- \sigma_j^-). \quad (3.6)$$

The generating function (3.2) can then be expressed as

$$\begin{aligned} \Omega_{\Psi_0}(\lambda) &= \frac{1}{L} \frac{\partial}{\partial x} \Big|_{x=\lambda} \langle \Psi_0 | \tau(i+x) \tau^{-1}(i+\lambda) | \Psi_0 \rangle \\ &\sim \frac{1}{L} \frac{\partial}{\partial x} \Big|_{x=\lambda} \text{Sp} \langle \Psi_0 | V_L(x, \lambda) \dots V_1(x, \lambda) | \Psi_0 \rangle, \end{aligned} \quad (3.7)$$

where $V_n(x, \lambda)$ are 4×4 matrices with entries $(V_n(x, \lambda))_{cd}^{ab}$ that are operators acting on the two-dimensional quantum space on site n

$$(V_n(x, \lambda))_{cd}^{ab} = (L_n(x))^{ab} (M_n(\lambda))^{cd}. \quad (3.8)$$

In this notation $V_L(x, \lambda) \dots V_1(x, \lambda)$ is a regular product of 4×4 matrices and Sp denotes the usual trace for 4×4 matrices. Let us now assume that $|\Psi_0\rangle$ is a product state

$$|\Psi_0\rangle = \otimes_{j=1}^L |\Psi_0^{(j)}\rangle. \quad (3.9)$$

Then Ω_{Ψ_0} can be written as

$$\Omega_{\Psi_0}(\lambda) \sim \frac{1}{L} \frac{\partial}{\partial x} \Big|_{x=\lambda} \text{Sp} \left[\prod_{j=1}^L U_j(x, \lambda) \right], \quad (3.10)$$

where

$$U_j(x, \lambda) = \langle \Psi_0^{(j)} | V_j(x, \lambda) | \Psi_0^{(j)} \rangle. \quad (3.11)$$

We now discuss how to implement the above programme for some explicit examples.

A. Quench from $|x, \uparrow\rangle$

Our first example is the product state

$$|x, \uparrow\rangle = \otimes_{j=1}^L \frac{|\uparrow\rangle_j + |\downarrow\rangle_j}{\sqrt{2}}. \quad (3.12)$$

This corresponds to all spins pointing in the x-direction. This initial state corresponds to a quantum quench in the XXZ-chain with an applied *transverse* magnetic field

$$H(h) = \frac{1}{4} \sum_{\ell=1}^L \sigma_\ell^x \sigma_{\ell+1}^x + \sigma_\ell^y \sigma_{\ell+1}^y + \Delta (\sigma_\ell^z \sigma_{\ell+1}^z - 1) - \frac{h}{2} \sum_{j=1}^L \sigma_j^x. \quad (3.13)$$

Preparing the system in the ground state of $H(\infty)$ gives the initial state (3.12), and the quench is to the integrable zero-field Hamiltonian $H(0)$. Using translational invariance we have

$$\Omega_{x, \uparrow}(\lambda) \sim \frac{1}{L} \frac{\partial}{\partial x} \Big|_{x=\lambda} \text{Sp} \left[(U(x, \lambda))^L \right]. \quad (3.14)$$

Denoting the largest eigenvalue of $U(x, \lambda)$ by $\mu_{\max}(x, \lambda)$, this gives

$$\Omega_{x,\uparrow}(\lambda) \sim \frac{1}{L} \frac{\partial}{\partial x} \Big|_{x=\lambda} (\mu_{\max}(x, \lambda))^L = \frac{\partial}{\partial x} \Big|_{x=\lambda} \mu_{\max}(x, \lambda). \quad (3.15)$$

In the last step we have used that $\mu_{\max}(\lambda, \lambda) = 1$. The matrix $U(x, \lambda)$ is readily calculated and its largest eigenvalue, which for small x, λ is very close to 1, is calculated using Mathematica. This results in

$$\Omega_{x,\uparrow}(\lambda) = \frac{-i\gamma \sin(\gamma)}{2 + 2 \cosh(\gamma\lambda) + 4 \cos(\gamma)}. \quad (3.16)$$

Matching the power-series expansion around $\lambda = 0$ with (3.2) gives

$$\lim_{N \rightarrow \infty} \frac{\langle x, \uparrow | H^{(k)} | x, \uparrow \rangle}{N} = \frac{1 - \Delta}{4} \frac{\partial^{k-1}}{\partial x^{k-1}} \Big|_{x=0} \frac{1 + \Delta}{\cosh^2(\sqrt{1 - \Delta^2}x/2) + \Delta}. \quad (3.17)$$

The results for $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$ agree with Ref. [48].

B. Quench from the Néel state

We next consider the Néel state

$$|\text{Néel}\rangle = \otimes_{j=1}^{L/2} [|\uparrow\rangle_{2j-1} \otimes |\downarrow\rangle_{2j}]. \quad (3.18)$$

This would be the initial state for a quantum quench from the ground state of an initial XXZ Hamiltonian with $\Delta = +\infty$ to a final Hamiltonian with finite Δ . Using translational invariance by two sites, our expression (3.10) for the generating function becomes

$$\Omega_{\text{Néel}}(\lambda) \sim \frac{1}{L} \frac{\partial}{\partial x} \Big|_{x=\lambda} \text{Sp} \left[(U_1(x, \lambda) U_2(x, \lambda))^{L/2} \right], \quad (3.19)$$

where

$$\begin{aligned} U_1(x, \lambda) &= {}_1\langle \uparrow | V_1(x, \lambda) | \uparrow \rangle_1, \\ U_2(x, \lambda) &= {}_2\langle \downarrow | V_2(x, \lambda) | \downarrow \rangle_2. \end{aligned} \quad (3.20)$$

These two 4×4 matrices and the largest eigenvalue of $U_1(x, \lambda) U_2(x, \lambda)$ (for small x, λ) are readily calculated using Mathematica

$$\Omega_{\text{Néel}}(\lambda) = \frac{i\gamma}{2} \frac{\sin(2\gamma)}{2 \cosh(\gamma\lambda) - 1 - \cos(2\gamma)}. \quad (3.21)$$

Matching the power series expansion around $\lambda = 0$ with (2.3) then gives

$$\lim_{L \rightarrow \infty} \frac{\langle \text{Néel} | H^{(k)} | \text{Néel} \rangle}{L} = -\frac{\Delta}{2} \frac{\partial^{k-1}}{\partial x^{k-1}} \Big|_{x=0} \frac{1 - \Delta^2}{\cosh(\sqrt{1 - \Delta^2}x) - \Delta^2}. \quad (3.22)$$

C. More general initial states

In principle our method can accommodate more complicated initial states of matrix-product form

$$|\Psi_0\rangle = \tilde{\text{Tr}} \left[\otimes_{j=1}^L A_j \right], \quad (3.23)$$

where A_j is an $m \times m$ matrix with entries that are quantum states on site j , and $\tilde{\text{Tr}}$ is the trace over the m -dimensional matrix space. Considering such states is however beyond the scope of the present paper.

IV. GENERALIZED GIBBS ENSEMBLE AND QUANTUM TRANSFER MATRIX

Combining (1.4) with (2.3), we can express the density matrix of the truncated generalized Gibbs ensembles as

$$\rho_{\text{tGGE}}^{(y)} = \frac{1}{Z_{\text{tGGE}}^{(y)}} e^{-i \sum_{j=1}^y \lambda_j \left(\frac{\sin \gamma}{\gamma} \frac{d}{dy} \right)^j \log \tau(i+y)} \Big|_{y=0}. \quad (4.1)$$

This density matrix can be analyzed by the quantum transfer matrix approach⁴⁵. Following the analysis of Klümper and Sakai for computing the thermal conductivity at finite temperature in Ref. [49], we introduce inhomogeneities in the transfer matrix and define the ensemble

$$\rho_{\{u_{1;N}, \dots, u_{N;N}\}} = \frac{\lim_{N \rightarrow \infty} (\tau^{-1}(i) \tau(i + 2u_{1;N}/\gamma)) \cdots (\tau^{-1}(i) \tau(i + 2u_{N;N}/\gamma))}{Z_{\{u_{1;N}, \dots, u_{N;N}\}}}. \quad (4.2)$$

As transfer matrices with different spectral parameters commute $[\tau(\lambda_1), \tau(\lambda_2)] = 0$, we have

$$\rho_{\{u_{1;N}, \dots, u_{N;N}\}} = \frac{\lim_{N \rightarrow \infty} e^{\sum_{i=1}^N \log \tau(i + 2u_{i;N}/\gamma) - \log \tau(i)}}{Z_{\{u_{1;N}, \dots, u_{N;N}\}}}. \quad (4.3)$$

In order to achieve

$$\rho_{\{u_{1;N}, \dots, u_{N;N}\}} = \rho_{\text{tGGE}}^{(y)}, \quad (4.4)$$

we need to choose the inhomogeneities $u_{j;N}$ such that

$$\lim_{N \rightarrow \infty} \sum_{i=1}^N \log \tau(i + 2u_{i;N}/\gamma) - \log \tau(i) \stackrel{!}{=} -i \sum_{j=1}^y \lambda_j \left(\frac{\sin \gamma}{\gamma} \frac{d}{dy} \right)^j \log \tau(i+y) \Big|_{y=0}. \quad (4.5)$$

A sufficient condition for Eq. (4.5) to hold is that the spectral parameters $u_{j;N}$ satisfy

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N [f(i + 2u_{j;N}/\gamma) - f(i)] = -i \sum_{j=1}^y \lambda_j \left(\frac{\sin \gamma}{\gamma} \frac{d}{dy} \right)^j f(i+y) \Big|_{y=0} \quad (4.6)$$

for any function $f(y)$ analytic at $y = i$. We do not have to require further properties if we can find a solution such that

$$\lim_{N \rightarrow \infty} u_{j;N} = 0. \quad (4.7)$$

By series expanding f about zero, Eq. (4.6) can be rewritten as

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N u_{j;N}^\ell = \begin{cases} -i \left(\frac{\sin \gamma}{2} \right)^\ell \ell! \lambda_\ell & \text{if } \ell \leq y \\ 0 & \text{if } \ell > y. \end{cases} \quad (4.8)$$

A solution of (4.8) satisfying (4.7) is given by

$$u_{my+j;N} = u_{j;N}^{(y)} = \sum_{n=1}^y w_n^{(y)} \frac{e^{2\pi i j n / y}}{N^{n/y}} \quad j = 1, \dots, y \quad m = 0, \dots, \lfloor N/y \rfloor - 1, \quad (4.9)$$

provided that

$$\sum_{\substack{n_1, \dots, n_\ell=1 \\ \sum_{i=1}^\ell n_i = y}}^{y+1-\ell} w_{n_1}^{(y)} \cdots w_{n_\ell}^{(y)} = -i \left(\frac{\sin \gamma}{2} \right)^\ell \ell! \lambda_\ell. \quad (4.10)$$

Having represented the density matrix of our truncated GGE in terms of an inhomogeneous transfer matrix, we wish to generalize the quantum transfer matrix approach for finite temperature correlation functions [50] in order to determine expectation values of local operators in the (truncated) GGE. One can show that

$$Z_{\text{GGE}} \rho_{\text{GGE}} = \lim_{N \rightarrow \infty} \text{Tr}_{\vec{1}, \dots, \vec{N}} \left[T_1^{QTM}(0) \cdots T_L^{QTM}(0) \right], \quad (4.11)$$

where the monodromy matrix T_n^{QTM} is given by

$$T^{QTM}(x) = \begin{pmatrix} A(\gamma x/2) & B(\gamma x/2) \\ C(\gamma x/2) & D(\gamma x/2) \end{pmatrix} = \prod_{i=0}^{N-1} L_{2N-i}(x) M_{2N-1-i} \left(\frac{2u_{i;N}}{\gamma} - x \right), \quad (4.12)$$

and the matrices L and M are defined in Eqs (2.5)(3.6). In the antiferromagnetic regime $\Delta > 1$ it is customary to define $\Delta = \cosh \eta$, corresponding to $\eta = i\gamma$, and change variables as follows:

$$x = \frac{2}{\gamma} \lambda. \quad (4.13)$$

$T^{QTM}(2\lambda/\gamma)$ acts as an upper triangular matrix on the vector

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (4.14)$$

and we have

$$C(\lambda) |0\rangle = 0, \quad A(\lambda) |0\rangle = a(\lambda) |0\rangle, \quad D(\lambda) |0\rangle = d(\lambda) |0\rangle, \quad (4.15)$$

with

$$\begin{aligned} a(\lambda) &= \left[\prod_{j=1}^y \frac{\sinh(\lambda - u_{j;N}^{(y)})}{\sinh(\lambda - u_{j;N}^{(y)} - \eta)} \right]^{N/y}, \\ d(\lambda) &= \left(\frac{\sinh \lambda}{\sinh(\lambda + \eta)} \right)^N. \end{aligned} \quad (4.16)$$

The Bethe ansatz equations for the eigenvalues of the quantum transfer matrix take the form

$$\frac{a(w_j)}{d(w_j)} = \prod_{\substack{k=1 \\ k \neq j}}^M \frac{\sinh(w_j - w_k + \eta)}{\sinh(w_j - w_k - \eta)}. \quad (4.17)$$

We then introduce the auxiliary function

$$\mathbf{a}(\lambda) = \frac{d(\lambda)}{a(\lambda)} \prod_{k=1}^N \frac{\sinh(\lambda - v_k + \eta)}{\sinh(\lambda - v_k - \eta)}, \quad (4.18)$$

where v_1, \dots, v_N is the solution of the Bethe ansatz equations corresponding to the largest eigenvalue of the quantum transfer matrix $t^{QTM}(x) = \text{Tr}[T^{QTM}(x)]$. For $\Delta > 1$ the function $1 + \mathbf{a}(\lambda)$ has simple zeros at $\lambda = v_j$ (on the imaginary axis) and y poles of order N/y at $\lambda = u_{j;N}^{(y)}$ inside the rectangle Q defined by $|\text{Re}\lambda| < \eta/2$ and $|\text{Im}\lambda| < \pi/2$. Using these analytic properties we obtain the following integral equation for $\mathbf{a}(\lambda)$

$$\log \mathbf{a}(\lambda) = \lim_{N \rightarrow \infty} \frac{N}{y} \sum_{j=1}^y \log \left[\frac{\sinh(\lambda + \eta - u_{j;N}^{(y)}) \sinh(\lambda)}{\sinh(\lambda - u_{j;N}^{(y)}) \sinh(\lambda + \eta)} \right] - \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{\sinh(2\eta) \log(1 + \mathbf{a}(\omega))}{\sinh(\lambda - \omega + \eta) \sinh(\lambda - \omega - \eta)}, \quad (4.19)$$

where \mathcal{C} is a rectangular contour with edges parallel to the real axis at $\pm i\pi/2$ and to the imaginary axis at $\pm\gamma$ where $0 < \gamma < \frac{\eta}{2}$. We may use Eq. (4.6) to replace the inhomogeneities by the Lagrange multipliers λ_j specifying the truncated GGE

$$\lim_{N \rightarrow \infty} \frac{N}{y} \sum_{j=1}^y \log \left[\frac{\sinh(\lambda + \eta - u_{j;N}^{(y)}) \sinh(\lambda)}{\sinh(\lambda - u_{j;N}^{(y)}) \sinh(\lambda + \eta)} \right] = F \left[i \frac{\sinh \eta}{2} \frac{d}{d\lambda} \right] \left(\frac{\sinh \eta}{\sinh \lambda \sinh(\lambda + \eta)} \right), \quad (4.20)$$

where

$$F[x] = -\frac{\sinh \eta}{2} \sum_{j=0}^{y-1} \lambda_{j+1}^{(y)} x^j. \quad (4.21)$$

Finally we have

$$\log \mathbf{a}(\lambda) = F \left[i \frac{\sinh \eta}{2} \frac{d}{d\lambda} \right] \left(\frac{\sinh \eta}{\sinh \lambda \sinh(\lambda + \eta)} \right) - \oint_C \frac{d\omega}{2\pi i} \frac{\sinh(2\eta) \log(1 + \mathbf{a}(\omega))}{\sinh(\lambda - \omega + \eta) \sinh(\lambda - \omega - \eta)}. \quad (4.22)$$

We stress that this equation is valid only if the expectation value of S^z is zero. For non-vanishing $\langle \Psi_0 | S^z | \Psi_0 \rangle$ one needs to add a constant to the driving term, which corresponds to the Lagrange multiplier of the total spin in the z direction (see *e.g.* Ref. [50])

$$\log \mathbf{a}(\lambda) = h + F \left[i \frac{\sinh \eta}{2} \frac{d}{d\lambda} \right] \left(\frac{\sinh \eta}{\sinh \lambda \sinh(\lambda + \eta)} \right) - \oint_C \frac{d\omega}{2\pi i} \frac{\sinh(2\eta) \log(1 + \mathbf{a}(\omega))}{\sinh(\lambda - \omega + \eta) \sinh(\lambda - \omega - \eta)}. \quad (4.23)$$

For the sake of simplicity we will restrict our analysis to the $h = 0$ case in the following and return to the $h \neq 0$ in section VIII.

The inverse function $\bar{\mathbf{a}}(\lambda) \equiv 1/\mathbf{a}(\lambda)$ fulfils the following integral equation

$$\log \bar{\mathbf{a}}(\lambda) = F \left[i \frac{\sinh \eta}{2} \frac{d}{d\lambda} \right] \left(\frac{\sinh \eta}{\sinh(\lambda) \sinh(\lambda - \eta)} \right) + \oint_C \frac{d\omega}{2\pi i} \frac{\sinh(2\eta) \log(1 + \bar{\mathbf{a}}(\omega))}{\sinh(\lambda - \omega + \eta) \sinh(\lambda - \omega - \eta)}. \quad (4.24)$$

Similarly to the thermal case, the Lagrange multipliers enter into the calculation of correlation functions only through the auxiliary functions $\mathbf{a}(\lambda)$, $\bar{\mathbf{a}}(\lambda)$.

A. Thermodynamic Properties

Thermodynamic properties are completely determined by the largest eigenvalue $\Lambda_0(0)$ of the quantum transfer matrix. The logarithm of $\Lambda_0(0)$ is given by⁵⁰

$$\log \Lambda_0(0) \equiv \lim_{L \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\log Z_{N,L}}{L} = \oint_C \frac{d\omega}{2\pi i} \frac{\sinh \eta \log(1 + \mathbf{a}(\omega))}{\sinh \omega \sinh(\omega + \eta)} = - \oint_C \frac{d\omega}{2\pi i} \frac{\sinh \eta \log(1 + \bar{\mathbf{a}}(\omega))}{\sinh \omega \sinh(\omega - \eta)}. \quad (4.25)$$

The expectation values of the local conservation laws can then be expressed in the form

$$\frac{1}{L} \langle H^{(j)} \rangle = - \frac{\partial \log \Lambda_0(0)}{\partial \lambda_j^{(y)}} = - \oint_C \frac{d\omega}{2\pi i} \frac{\sinh \eta \partial_{\lambda_j} \mathbf{a}(\omega)}{\sinh \omega \sinh(\omega + \eta) (1 + \mathbf{a}(\omega))} = \oint_C \frac{d\omega}{2\pi i} \frac{\sinh \eta (\partial_{\lambda_j} \log \bar{\mathbf{a}}(\omega))}{\sinh \omega \sinh(\omega - \eta) (1 + \mathbf{a}(\omega))}. \quad (4.26)$$

Using (4.24), we obtain

$$\partial_{\lambda_j^{(y)}} \log \bar{\mathbf{a}}(\lambda) = - \left(i \frac{\sinh \eta}{2} \right)^{j-1} \partial_{\lambda}^{j-1} \left(\frac{\sinh^2 \eta}{2 \sinh(\lambda) \sinh(\lambda - \eta)} \right) + \oint_C \frac{d\omega}{2\pi i} \frac{\sinh(2\eta) (\partial_{\lambda_j^{(y)}} \log \bar{\mathbf{a}}(\omega))}{\sinh(\lambda - \omega + \eta) \sinh(\lambda - \omega - \eta) (1 + \mathbf{a}(\omega))}. \quad (4.27)$$

The solution to this integral equation is conveniently expressed in terms of an auxiliary function G

$$\partial_{\lambda_j^{(y)}} \log \bar{\mathbf{a}}(\lambda) = -i \left(-i \frac{\sinh \eta}{2} \right)^j \partial_{\mu}^{j-1} G(\lambda, \mu; 0) \Big|_{\mu=0}, \quad (4.28)$$

where

$$\begin{aligned} G(\lambda, \mu; \alpha) &= -\coth(\lambda - \mu) + e^{\alpha\eta} \coth(\lambda - \mu - \eta) + \oint_C \frac{d\omega}{2\pi i} \frac{G(\omega, \mu; \alpha)}{1 + \mathbf{a}(\omega)} K(\lambda - \omega; \alpha), \\ K(\lambda; \alpha) &= e^{\alpha\eta} \coth(\lambda - \eta) - e^{-\alpha\eta} \coth(\lambda + \eta). \end{aligned} \quad (4.29)$$

Substituting this back into (4.26), we obtain

$$\frac{1}{L} \langle H_j \rangle = -i \left(\frac{-i \sinh \eta}{2} \right)^j \partial_{\mu}^{j-1} \oint_C \frac{d\omega}{2\pi i} \frac{\sinh \eta G(\omega, \mu; 0)}{\sinh \omega \sinh(\omega - \eta) (1 + \mathbf{a}(\omega))} \Big|_{\mu=0}. \quad (4.30)$$

For the case of the full generalized Gibbs ensemble, i.e. the limit $y \rightarrow \infty$ of the truncated GGE considered above, we may lift the relationship (4.30) to the level of the generating function $\Omega_{\Psi_0}(\mu)$ defined in (3.2)

$$\partial_{\mu}^{j-1} \left[\oint_C \frac{d\omega}{2\pi i} \frac{\sinh \eta G(\omega, \mu; 0)}{\sinh \omega \sinh(\omega - \eta) (1 + \mathbf{a}(\omega))} + \left(\frac{2i}{\eta} \right)^j \Omega_{\Psi_0}(\mu) \right] \Big|_{\mu=0} = 0, \quad j = 1, 2, \dots \quad (4.31)$$

This can be rewritten in a more compact form as

$$\oint_C \frac{d\omega}{2\pi i} \frac{\sinh \eta G(\omega, \mu; 0)}{\sinh \omega \sinh(\omega - \eta)(1 + \mathbf{a}(\omega))} = -\frac{2i}{\eta} \Omega_{\Psi_0} \left(\frac{2i\mu}{\eta} \right). \quad (4.32)$$

In passing we note that the magnetization can be computed analogously by taking the derivative with respect to the magnetic field of the integral equation (4.23), which gives

$$\oint_C \frac{d\omega}{2\pi i} \frac{G(\omega, 0; 0)}{1 + \mathbf{a}(\omega)} = -\frac{\langle \sigma^z \rangle + 1}{2}. \quad (4.33)$$

To summarize the result of this section: in the thermodynamic limit the GGE can be formulated in terms of a quantum transfer matrix, whose largest eigenvalue is given in terms of the functions \mathbf{a} and $\bar{\mathbf{a}}$, which in turn are defined through the nonlinear integral equations (4.22) and (4.24) respectively. The expectation values of the local integrals of motion can then be expressed in terms of \mathbf{a} and the auxiliary function $G(\lambda, \mu; 0)$ defined in (4.29). In order to proceed for the case of a truncated GGE, it is necessary to solve our system of integral equations subject to the constraints (4.30). Carrying out such a computation entails calculating the Lagrange multipliers $\lambda_j^{(y)}$. On the other hand, we are ultimately interested in the full GGE itself. In this case it is possible to avoid having to determine the Lagrange multipliers $\lambda_j^{(y)}$, as we will show next.

V. ELIMINATING THE LAGRANGE MULTIPLIERS

In order to proceed, it is convenient to switch to the “ $\mathbf{b}\bar{\mathbf{b}}$ -formulation” of the integral equations by defining

$$\begin{aligned} \mathbf{b}(x) &= \mathbf{a}\left(ix^+ + \frac{\eta}{2}\right), \quad \bar{\mathbf{b}}(x) = \bar{\mathbf{a}}\left(ix^- - \frac{\eta}{2}\right), \\ g_\mu^\pm(x) &= \pm G\left(ix^\pm \pm \frac{\eta}{2}, i\mu; 0\right), \quad x^\pm = x \pm i\epsilon. \end{aligned} \quad (5.1)$$

The functions (5.1) can be shown to obey the integral equations

$$\begin{aligned} \log \mathbf{b}(x) &= F\left[\frac{\sin \eta}{2} \frac{\partial}{\partial x}\right] d(x) + [k * \log(1 + \mathbf{b})](x) - [k_- * \log(1 + \bar{\mathbf{b}})](x), \\ \log \bar{\mathbf{b}}(x) &= F\left[\frac{\sin \eta}{2} \frac{\partial}{\partial x}\right] d(x) + [k * \log(1 + \bar{\mathbf{b}})](x) - [k_+ * \log(1 + \mathbf{b})](x), \\ g_\mu^+(x) &= -d(x - \mu) + \left[k * \frac{g_\mu^+}{1 + \mathbf{b}^{-1}}\right](x) - \left[k_- * \frac{g_\mu^-}{1 + \bar{\mathbf{b}}^{-1}}\right](x), \\ g_\mu^-(x) &= -d(x - \mu) + \left[k * \frac{g_\mu^-}{1 + \bar{\mathbf{b}}^{-1}}\right](x) - \left[k_+ * \frac{g_\mu^+}{1 + \mathbf{b}^{-1}}\right](x) \end{aligned} \quad (5.2)$$

where $[f_1 * f_2](x) = \int_{-\pi/2}^{\pi/2} \frac{dx}{\pi} f_1(x - y) f_2(y)$ and

$$d(x) = \sum_{n=-\infty}^{\infty} \frac{e^{2inx}}{\cosh(\eta n)}, \quad k(x) = \sum_{n=-\infty}^{\infty} \frac{e^{2inx}}{e^{2\eta|n|} + 1}, \quad k_\pm(x) = k(x^\mp \pm i\eta). \quad (5.3)$$

As shown in Refs [47], $d(x)$ and $k(x)$ can be expressed in terms of special functions. In terms of the new variables, equation (4.32) is rewritten as

$$-\int_{-\pi/2}^{\pi/2} \frac{dx}{\pi} d(x) \left(\frac{g_\mu^+(x)}{1 + \mathbf{b}^{-1}(x)} + \frac{g_\mu^-(x)}{1 + \bar{\mathbf{b}}^{-1}(x)} \right) = 4k(\mu) + \frac{4i}{\eta} \Omega_{\Psi_0}(-2\mu/\eta). \quad (5.4)$$

The Lagrange multipliers λ_j are just parameters of the integral equations, which are fixed by Eq. (5.4). Quite surprisingly, we can remove the explicit dependence on λ_j by taking the difference between the first two equations of

system (5.2), i.e. considering the system of integral equations

$$\begin{aligned}
\log \mathfrak{b}(x) - \log \bar{\mathfrak{b}}(x) &= [(k_+ + k) * \log(1 + \mathfrak{b})](x) - [(k_- + k) * \log(1 + \bar{\mathfrak{b}})](x) , \\
g_\mu^+(x) &= -d(x - \mu) + \left[k * \frac{g_\mu^+}{1 + \mathfrak{b}^{-1}} \right](x) - \left[k_- * \frac{g_\mu^-}{1 + \bar{\mathfrak{b}}^{-1}} \right](x) , \\
g_\mu^-(x) &= -d(x - \mu) + \left[k * \frac{g_\mu^-}{1 + \bar{\mathfrak{b}}^{-1}} \right](x) - \left[k_+ * \frac{g_\mu^+}{1 + \mathfrak{b}^{-1}} \right](x) , \\
4k(\mu) + \frac{4i}{\eta} \Omega_{\Psi_0}(-2\mu/\eta) &= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\pi} d(x) \left(\frac{g_\mu^+(x)}{1 + \mathfrak{b}^{-1}(x)} + \frac{g_\mu^-(x)}{1 + \bar{\mathfrak{b}}^{-1}(x)} \right) .
\end{aligned} \tag{5.5}$$

In general (5.5) have to be solved numerically by iteration. In addition, from the the same equations it follows

$$F[in \sinh \eta] = \frac{e^{-\eta n} [\log \mathfrak{b}]_n + e^{\eta n} [\log \bar{\mathfrak{b}}]_n}{2} , \tag{5.6}$$

where

$$[f]_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\pi} e^{-2inx} f(x) . \tag{5.7}$$

By setting $n = 0$ in (5.6), one obtains the inverse temperature (the Lagrange multiplier of the Hamiltonian). The other Lagrange multipliers are more difficult to compute, as it is not always easy to identify the Taylor coefficients of $F[x]$ from the values of the Fourier coefficients on the right hand side (the case considered in Section VIA is an example). This suggests that solving the system (5.5) is less demanding than working out (5.2) by recursively computing the Lagrange multipliers.

VI. PERTURBATIVE EXPANSION

We now consider examples, in which (5.5) can be solved through an expansion that can be carried out analytically. Importantly, this shows that it is indeed possible to completely specify the quantum transfer matrix describing the GGE without having to calculate the Lagrange multipliers λ_j .

If we consider the expectation value on the ground state of the model, it is known that

$$\Omega_{GS}(-2\mu/\eta) = i\eta k(\mu) , \tag{6.1}$$

where $k(\mu)$ is given in (5.3). Eqn (6.1) can be inferred directly from (5.5), (5.6) by considering the case $\mathfrak{b}(x) = \epsilon(1 + \mathcal{O}(\epsilon^2))$ and $\bar{\mathfrak{b}}(x) = \epsilon(1 + \mathcal{O}(\epsilon^2))$ in the limit $\epsilon \rightarrow 0$. This is equivalent to the zero temperature limit in the Gibbs ensemble, which in the $\Delta > 1$ case clearly tends to the thermodynamic ground state of the model (see also Ref. [50]).

This observation opens the door for carrying out a perturbative expansion of the system (5.5) in the limit of a “small” quench. We define functions $\rho(x)$ and $\zeta(x)$ by

$$\begin{aligned}
\frac{1}{1 + \mathfrak{b}^{-1}(x)} &= e^{\zeta(x)/2} \rho(x) , \\
\frac{1}{1 + \bar{\mathfrak{b}}^{-1}(x)} &= e^{-\zeta(x)/2} \rho(x) .
\end{aligned} \tag{6.2}$$

In terms of these new variables, (5.5) are rewritten as

$$\begin{aligned}
\zeta(x) &= -[(k + k_+) * \log(1 - e^{\zeta/2} \rho)](x) + [(k + k_-) * \log(1 - e^{-\zeta/2} \rho)](x) + \log \left(\frac{1 - e^{\zeta(x)/2} \rho(x)}{1 - e^{-\zeta(x)/2} \rho(x)} \right) , \\
g_\mu^+(x) &= -d(x - \mu) + \left[k * (\rho e^{\zeta/2} g_\mu^+) \right](x) - \left[k_- * (\rho e^{-\zeta/2} g_\mu^-) \right](x) , \\
g_\mu^-(x) &= -d(x - \mu) + \left[k * (\rho e^{-\zeta/2} g_\mu^-) \right](x) - \left[k_+ * (\rho e^{\zeta/2} g_\mu^+) \right](x) , \\
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{4\pi} d(x) \rho(x) \left[e^{\zeta(x)/2} g_\mu^+(x) + e^{-\zeta(x)/2} g_\mu^-(x) \right] &= -\frac{i}{\eta} \Omega_{\Psi_0}(-2\mu/\eta) - k(\mu) .
\end{aligned} \tag{6.3}$$

The definition of a “small” quench is one for which $|\rho(x)| \ll 1$. In this case we may solve (6.3) by iteration. At lowest order we have

$$\rho(x) \sim \rho^{(1)}(x), \quad \zeta(x) \sim 0, \quad g_\mu^\pm(x) \sim -d(x - \mu). \quad (6.4)$$

At this order, the last equation of (5.5) reads

$$[d * (\rho^{(1)} d)](\mu) = 2k(\mu) + \frac{2i}{\eta} \Omega_{\Psi_0}(-2\mu/\eta). \quad (6.5)$$

This can be solved by Fourier techniques. The n ’th Fourier coefficient is

$$\frac{[\rho^{(1)} d]_n}{\cosh(\eta n)} = \frac{2}{1 + e^{2\eta|n|}} + \frac{2i}{\eta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\mu}{\pi} e^{-2in\mu} \Omega_{\Psi_0}(-2\mu/\eta), \quad (6.6)$$

and going back to real space we obtain

$$\rho^{(1)}(x) = \frac{\frac{i}{\eta} \Omega_{\Psi_0}(-\frac{2\mu}{\eta} + i) + \frac{i}{\eta} \Omega_{\Psi_0}(-\frac{2\mu}{\eta} - i) + K_\eta(x)}{d(x)}, \quad (6.7)$$

where

$$K_\eta(x) = \frac{\sinh \eta}{\cosh \eta - \cos(2x)}. \quad (6.8)$$

In the next order of iteration one replaces $\rho(x)$ with $\rho^{(1)}(x)$ in the equations defining $\zeta(x)$ and $g_\mu^\pm(x)$, and solving the resulting system gives the improved result $\rho^{(2)}(x)$

$$\rho^{(2)}(x) = \rho^{(1)}(x) \left\{ 1 - \frac{1}{d(x)} \left[\left(k - \frac{1}{4} K_{2\eta} \right) * (\rho^{(1)} d) \right](x) \right\}. \quad (6.9)$$

The parameter

$$\kappa \equiv \max_x \left| \frac{1}{d(x)} \left[\left(k - \frac{1}{4} K_{2\eta} \right) * (\rho^{(1)} d) \right](x) \right| \quad (6.10)$$

gives an estimate for the accuracy of the first order approximation. We will show in Section VII that for a small quench the entire dependence on the initial state is encoded in $\rho^{(1)}(x)$. Short-range spin-spin correlators are then expressed in terms of integrals involving $\rho^{(1)}(x)$ and other known functions, which depend *only* on the anisotropy parameter Δ .

A. Example: quench from the Néel state ($\Delta_0 = +\infty$) to finite, large Δ

In this section we consider a quench from the Néel state, for which $\Omega_{\text{Néel}}(\mu)$ is given in (3.21). The lowest order result for $\rho^{(1)}(x)$ (6.7) is given by

$$\rho^{(1)}(x) = \frac{\sin^2(2x) \tanh^2(\eta) \sinh(\eta)}{[\cosh(\eta) - \cos(2x)][(\cosh(\eta) - \cos(2x))^2 + \tanh^2(\eta) \sin^2(2x)]d(x)} \xrightarrow{1 \ll \eta} \frac{\sin^2(2x)}{\cosh^2(\eta)}. \quad (6.11)$$

This is proportional to $1/\Delta^2$ and for sufficiently large Δ indeed small.

VII. CORRELATION FUNCTIONS

In the last years there has been tremendous progress in the calculation of equal time correlation functions in the spin-1/2 XXZ chain^{46,47,50,51}. These results can be applied also to the generalized Gibbs ensemble of interest here. In particular, we may use the explicit expressions for short-distance correlators given in Ref. [47], which involve three functions $\varphi(\mu)$, $\omega(\mu_1, \mu_2)$ and $\omega'(\mu_1, \mu_2)$. Examples are

$$\langle \sigma_1^z \sigma_2^z \rangle = \text{cth}(\eta) \omega + \frac{\omega'_x}{\eta}, \quad \langle \sigma_1^x \sigma_2^x \rangle = -\frac{\omega}{2 \sinh(\eta)} - \frac{\cosh(\eta) \omega'_x}{2\eta}, \quad (7.1)$$

where $\omega = \omega(0, 0)$, $\omega'_x = \partial_x \omega'(x, y)|_{x,y=0}$. The corresponding expressions for the GGE are identical, but the functions $\varphi(\mu)$, $\omega(\mu_1, \mu_2)$, $\omega'(\mu_1, \mu_2)$ are different. Following the finite temperature case, we define functions

$$g_\mu^\pm(x) = \pm \partial_\alpha G\left(ix^\pm \pm \frac{\eta}{2}, i\mu; \alpha\right)\Big|_{\alpha=0}. \quad (7.2)$$

It follows from Eq. (4.29) that $g_\mu^{\pm}(x)$ satisfy the integral equations

$$\begin{aligned} g_\mu^+(x) &= -\eta c_+(x - \mu) + \eta \left[\ell * \frac{g_\mu^+}{1 + \mathbf{b}^{-1}} \right](x) - \eta \left[\ell_- * \frac{g_\mu^-}{1 + \bar{\mathbf{b}}^{-1}} \right](x) + \left[\kappa * \frac{g^+}{1 + \mathbf{b}^{-1}} \right](x) - \left[\kappa_- * \frac{g'^-}{1 + \bar{\mathbf{b}}^{-1}} \right](x), \\ g_\mu^-(x) &= -\eta c_-(x - \mu) + \eta \left[\ell * \frac{g_\mu^-}{1 + \bar{\mathbf{b}}^{-1}} \right](x) - \eta \left[\ell_+ * \frac{g_\mu^+}{1 + \mathbf{b}^{-1}} \right](x) + \left[\kappa * \frac{g'^-}{1 + \bar{\mathbf{b}}^{-1}} \right](x) - \left[\kappa_+ * \frac{g'^+}{1 + \mathbf{b}^{-1}} \right](x), \end{aligned} \quad (7.3)$$

where

$$\ell(x) = \sum_{n=-\infty}^{\infty} \frac{\text{sgn}(n) e^{2inx}}{4 \cosh^2(\eta n)}, \quad \ell_\pm(x) = \ell(x^\mp \pm i\eta), \quad c_\pm(x) = \pm \sum_{n=-\infty}^{\infty} \frac{e^{\pm \eta n + 2inx}}{2 \cosh^2(\eta n)}. \quad (7.4)$$

The three functions entering the expressions for short-range correlators are then given by

$$\begin{aligned} \varphi(\mu) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{2\pi} \left(\frac{g_{-i\mu}^-(x)}{1 + \bar{\mathbf{b}}^{-1}(x)} - \frac{g_{-i\mu}^+(x)}{1 + \mathbf{b}^{-1}(x)} \right) \\ \omega(\mu_1, \mu_2) &= \omega_{(0)}(\mu_1, \mu_2) - \left[d * \left(\frac{g_{-i\mu_1}^+(x)}{1 + \mathbf{b}^{-1}(x)} + \frac{g_{-i\mu_1}^-(x)}{1 + \bar{\mathbf{b}}^{-1}(x)} \right) \right](-i\mu_2) \\ \omega'(\mu_1, \mu_2) &= \omega'_{(0)}(\mu_1, \mu_2) - \left[d * \left(\frac{g_{-i\mu_1}^+}{1 + \mathbf{b}^{-1}} + \frac{g_{-i\mu_1}^-}{1 + \bar{\mathbf{b}}^{-1}} \right) \right](-i\mu_2) - \eta \left[c_- * \frac{g_{-i\mu_1}^+}{1 + \mathbf{b}^{-1}} \right](-i\mu_2) - \eta \left[c_+ * \frac{g_{-i\mu_1}^-}{1 + \bar{\mathbf{b}}^{-1}} \right](-i\mu_2), \end{aligned} \quad (7.5)$$

where

$$\begin{aligned} \omega_{(0)}(\mu_1, \mu_2) &= -4k(i\mu_1 - i\mu_2) + K_{2\eta}(i\mu_1 - i\mu_2) \\ \omega'_{(0)}(\mu_1, \mu_2) &= -4\eta \ell(i\mu_1 - i\mu_2) + \eta K_{2(\mu_1 - \mu_2)}(i\eta). \end{aligned} \quad (7.6)$$

A. Perturbative results for “small” quenches

For small quenches as defined in section VI we may use the iterative solution of the integral equations for \mathbf{b} to obtain approximate expressions for the functions φ , ω and ω' . Using the parametrization (6.2) and the lowest order solution (6.7), we find the following expressions for the first order approximations to the functions (7.5)

$$\begin{aligned} \varphi_{(1)}(\mu) &= 0, \\ \omega_{(1)}(\mu_1, \mu_2) &= \omega_{(0)}(\mu_1, \mu_2) + 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\pi} d(x + i\mu_1) d(x + i\mu_2) \rho^{(1)}(x), \\ \omega'_{(1)}(\mu_1, \mu_2) &= \omega'_{(0)}(\mu_1, \mu_2) + \eta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\pi} d(x + i\mu_2) \rho^{(1)}(x) \bar{c}(x + i\mu_1) - \eta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\pi} \bar{c}(x + i\mu_2) \rho^{(1)}(x) d(x + i\mu_1), \end{aligned} \quad (7.7)$$

where we have defined $\bar{c}(x) = c_+(x) + c_-(x)$. We can determine the functions entering the expressions for short-distance correlation functions such as (7.1) by numerical integration of the appropriate partial derivatives of (7.7) at $(0, 0)$, once the function $\rho^{(1)}$ is known.

Results for a quench from the Néel state, where $\rho^{(1)}(x)$ is given by (6.11), are presented in Table I. We stress that these are not obtained through a numerical solution of the nonlinear integral equations; we have merely evaluated the analytic expressions (6.11)(7.7). To the best of our knowledge, these are the first closed-form expressions for correlators in the stationary state after a nontrivial interacting quench.

$\frac{\Delta=2}{(\kappa \sim 0.1)}$	ℓ	$\langle \sigma_1^z \sigma_{1+\ell}^z \rangle$	$\langle \sigma_1^x \sigma_{1+\ell}^x \rangle$	$\frac{\Delta=3}{(\kappa \sim 0.03)}$	ℓ	$\langle \sigma_1^z \sigma_{1+\ell}^z \rangle$	$\langle \sigma_1^x \sigma_{1+\ell}^x \rangle$
	1	-0.661371	-0.338629		1	-0.815293	-0.27706
	2	0.376573	0.056895		2	0.643582	0.039557
	3	-0.279034	-0.009872		3	-0.57034	-0.005788
$\frac{\Delta=4}{(\kappa \sim 0.01)}$	ℓ	$\langle \sigma_1^z \sigma_{1+\ell}^z \rangle$	$\langle \sigma_1^x \sigma_{1+\ell}^x \rangle$	$\frac{\Delta=5}{(\kappa \sim 0.008)}$	ℓ	$\langle \sigma_1^z \sigma_{1+\ell}^z \rangle$	$\langle \sigma_1^x \sigma_{1+\ell}^x \rangle$
	1	-0.887611	-0.224779		1	-0.925316	-0.186711
	2	0.779648	0.025859		2	0.852509	0.0177252
	3	-0.730205	-0.003019		3	-0.81808	-0.001699

TABLE I: The short-range correlators for various values of Δ after a quench from the Néel state ($\Delta_0 = +\infty$) at lowest order of the perturbative expansion; κ (6.10) is an estimate of the relative error for $\rho^{(1)}(x)$.

VIII. INITIAL STATES WITH NONZERO LONGITUDINAL MAGNETIZATION

If the initial state has a nonzero magnetization

$$m^z = \frac{1}{2L} \sum_{\ell} \langle \sigma_{\ell}^z \rangle \neq 0 \quad (8.1)$$

the integral equations (5.5) must be modified as follows

$$\begin{aligned}
& \log \mathbf{b}(x) - \log \bar{\mathbf{b}}(x) + h = [(k_+ + k) * \log(1 + \mathbf{b})](x) - [(k_- + k) * \log(1 + \bar{\mathbf{b}})](x) , \\
& g_{\mu}^{+}(x) = -d(x - \mu) + \left[k * \frac{g_{\mu}^{+}}{1 + \mathbf{b}^{-1}} \right](x) - \left[k_- * \frac{g_{\mu}^{-}}{1 + \bar{\mathbf{b}}^{-1}} \right](x) , \\
& g_{\mu}^{-}(x) = -d(x - \mu) + \left[k * \frac{g_{\mu}^{-}}{1 + \bar{\mathbf{b}}^{-1}} \right](x) - \left[k_+ * \frac{g_{\mu}^{+}}{1 + \mathbf{b}^{-1}} \right](x) , \\
& - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\pi} d(x) \left(\frac{g_{\mu}^{+}(x)}{1 + \mathbf{b}^{-1}(x)} + \frac{g_{\mu}^{-}(x)}{1 + \bar{\mathbf{b}}^{-1}(x)} \right) = 4k(\mu) + \frac{4i}{\eta} \Omega_{\Psi_0}(-2\mu/\eta) , \\
& \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\pi} \left(\frac{g_0^{+}(x)}{1 + \mathbf{b}^{-1}(x)} - \frac{g_0^{-}(x)}{1 + \bar{\mathbf{b}}^{-1}(x)} \right) = 4m^z ,
\end{aligned} \quad (8.2)$$

where h is the Lagrange multiplier of the total spin in the longitudinal direction.

For a small quench, for which the magnetization is close to zero, the right hand side of the last two equations is still small, so we can still use the approach developed in Section VI. The main difference is that at lowest order the function $\zeta(x)$ is now a nonzero constant. The modification is in fact trivial, and we easily obtain

$$\sinh(\zeta/2) \sim -\frac{2m^z}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\pi} d(x) \rho_0^{(1)}(x)} , \quad \rho^{(1)}(x) \sim \frac{\rho_0^{(1)}(x)}{\cosh(\zeta/2)} , \quad (8.3)$$

where $\rho_0^{(1)}(x)$ is the function (6.7). As a matter of fact, at the lowest order $\omega_{(1)}(\mu_1, \mu_2)$ and $\omega'_{(1)}(\mu_1, \mu_2)$ have the same form in terms of the generating function Ω_{Ψ_0} as for $m^z = 0$. The only difference is in $\varphi(\mu)$, which is now different from zero

$$\varphi_{(1)}(\mu) = \tanh(\zeta/2) [d * \rho_0^{(1)}](-i\mu) . \quad (8.4)$$

IX. SUMMARY AND CONCLUSIONS

We have considered the problem of the late time behaviour of short-distance spin-spin correlators in the spin-1/2 Heisenberg XXZ chain after certain quantum quenches. We focussed on the antiferromagnetically ordered regime of the XXZ chain and assumed that at infinite times a generalized Gibbs ensemble is reached. Following Ref. [49] we constructed quantum transfer matrix description of this ensemble, and derived a set of nonlinear integral equations that

describe the largest eigenvalue of the quantum transfer matrix. We emphasize that there is an important difference between our description of the stationary state after a quench and thermal ensembles for generalized integrable Hamiltonians of the kind considered in Ref. [49]: in our case the “input data” are not given by the Lagrange multipliers (inverse temperature, chemical potential, etc.), but instead fixed implicitly through the expectation values of local conservation laws.

Our main results are as follows.

- (i) We have presented a method for calculating the expectation values of local conservation laws in simple initial states (of product or matrix-product form). We obtained explicit analytic expressions for two cases of interest: the state with all spins aligned in the transverse direction (Section III A) and the Néel state (Section III B).
- (ii) We have shown that as long as the expectation values of local conservation laws are known explicitly, it is possible to avoid having to determine the Lagrange multipliers defining the density matrix of the generalized Gibbs ensemble. This substantially simplifies the calculation of short-distance correlation functions in the stationary state (Section V).
- (iii) We showed that for “small” quenches, as defined in the main text, analytic expressions for correlation functions can be obtained. The properties of the initial state enter through a single function (Section VI), which we determined analytically for quenches from the Néel state (Section VI A). Using the results of Ref. [47] for finite-temperature correlation functions in the XXZ-chain, we obtained expressions for short distance spin correlation functions. In particular, we obtained an analytic expression for short-range correlators after a quantum quench from the Néel state to an XXZ Hamiltonian with sufficiently large anisotropy parameter Δ (Section VII A). The next orders of the perturbation expansion in $1/\Delta^2$ are easily accessible as well.

Much work remains. For other initial states such as $|x, \uparrow\rangle$ considered in section III A, we need to solve the nonlinear integral equations numerically or implement an iterative analysis similar to that for the Néel state. Work on this is in progress. For quenches to the critical XXZ chain $|\Delta| \leq 1$ the role of the conservation laws discovered recently⁴⁴ needs to be clarified.

While this work was being written up, a preprint by B. Pozsgay appeared on the arXiv⁵², where the same problem is studied by means of a quantum transfer matrix formulation, that allows for the calculation of short-range correlators by using the corresponding finite-temperature results. The main differences to our work are the treatments of the initial state and of the Lagrange multipliers.

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